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## THESIS

SEARCH PLANNING UNDER  
INCOMPLETE INFORMATION USING  
STOCHASTIC OPTIMIZATION AND REGRESSION

by

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**SEARCH PLANNING UNDER INCOMPLETE INFORMATION  
USING STOCHASTIC OPTIMIZATION AND REGRESSION**

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# ABSTRACT

This thesis deals with a type of stochastic optimization problem where the decision maker does not have complete information concerning the objective function. Specifically, we consider a discrete time-and-space search optimization problem where we seek to find a moving target in an area of operations. There are two sources of uncertainty: the target location and the sensor performance. We formulate the objective function for this problem in terms of a risk measure of a parameterized random variable and consider three cases involving various degrees of knowledge about the sensor performance. In all cases, we consider both the expectation and superquantile risk measures. While the expectation results in an objective function representing the probability of missing the target, the superquantile gives rise to more conservative search plans that perform reasonably well even under exceptional circumstances. In the case of incomplete information about the distribution of the sensor performance, we approximate the random variable using a nonstandard regression that minimizes the error induced in some sense. We examine the cases in a series of numerical examples.

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# EXECUTIVE SUMMARY

In this thesis, we discuss a type of stochastic optimization problem where the decision maker does not have complete information concerning the objective function. Specifically, we consider a discrete time-and-space search optimization problem (SP) of routing search assets in order to detect a moving target, such as a drug smuggler, in an area of operations.

We assume that each searcher is equipped with one imperfect sensor, which is not subject to false-positive detections. The searchers are all alike, equipped with the same type of sensor, and they are allowed to divide their effort across multiple cells in arbitrarily small portions. The goal is to determine a search plan such that the probability of missing the target is minimized, without having the full knowledge of the two sources of uncertainty: the target location and the sensor performance. We assume that the target paths are probabilistically known via intelligence reports; for example from data acquired by AIS (Automatic Identification System). We use a Markov chain model to generate the paths with three stationary probabilities. We assume that the sensor performance depends on environmental conditions, and we use visibility to represent these conditions.

We formulate the objective function of SP in terms of a risk measure of a parameterized random variable and consider three cases involving various degrees of knowledge about the sensor performance. In all cases, we consider both the expectation and superquantile risk measures. While the expectation results in an objective function representing the probability of missing the target, the superquantile gives rise to more conservative search plans that perform reasonably well even under exceptional circumstances.

In the first two cases, we consider a random detection rate for the sensor performance with known probability distribution. In the third case, the distribution of the sensor performance is unknown and we approximate the corresponding random

variables by a linear combination of well-known factors using a nonstandard regression. Based on a table of observed sensor performances, that regression minimizes the error induced by the approximation in some sense. We compare the obtained regression coefficients with the ones resulting from a least-squares linear regression model. Using the approximations in the objective function of SP, we obtain an approximate problem that we solve for situations involving stochastic information about the factors in the regression model.

We examine the cases in a series of numerical examples. In all cases, the numerical results show that the plans obtained using superquantile as the risk measure spread the searchers over the area of operations, covering a larger area with smaller searcher fractions than when using expectation. These plans handle exceptional target paths more effectively. If the goal is to make sure that the probability of missing the target does not exceed a relatively high threshold, then the decision maker should rely on the superquantile at probability level  $\alpha$  as the risk measure. One example shows that if the goal is to avoid probabilities of missing the target of 70% or higher, then the decision maker should use superquantile with  $\alpha = 0.90$ . In that situation there is a 92% chance of obtaining better mission outcomes than the goal, while there is only 81% chance of getting better than the goal when using the expectation. This difference might be critical when choosing a search plan and it even is more significant for faster moving targets.

# I. INTRODUCTION

## A. MOTIVATION AND BACKGROUND

Stochastic optimization problems arise in a diverse spectrum of real-life situations, but they can be challenging to formulate and solve since they involve uncertainty and unknown parameters (see for example Shapiro et al., 2009; Wallace & Ziemba, 2005).

One application area is engineering design, where the goal is to minimize the cost of a design, which might involve a simple objective function and complicated chance constraints. For example, the design must be able to resist future unknown loads, material properties, and environmental conditions; see Arora and Wang (2005), Royset et al. (2006), and Rockafellar and Royset (2010) for formulations and Luedtke and Ahmed(2008), and Shapiro et al. (2009) for algorithms and properties of these constraints. Another area of application is financial engineering where studies show the importance of selecting a suitable risk measure (Artzner et al., 1999) in portfolio management. A particular risk measure is the superquantile (also called conditional value-at-risk), which is coherent (Rockafellar & Uryasev, 2002), and therefore may be suitable for use in many applications.

In military operations, it is common for leaders to make decisions without the full knowledge of future events such as enemy actions, environmental conditions, and asset availability. We refer to a combination of future events as a scenario. Due to advanced coordination and planning that may involve numerous operational and logistical units, leaders may have to ignore the possibility of changing their decision in the middle of the action. Hence, recourse may not be available after a scenario is revealed.

In this thesis, we focus on search for a target in an area of operations where uncertainty is prevalent. Specifically, we consider two types of uncertainty: sensor performance and target location. The sensor performance depends on several aspects,

e.g., operator experience, environmental conditions such as visibility, and target characteristics. Search problems of this kind arise in military operations as well as search and rescue operations.

While some stochastic problems involve simple objective functions and complicated chance constraints, our search problem has an objective function that is more complex since it is defined by a risk measure of a parametrized random variable. A risk measure maps a random variable to the extended real numbers. We focus on the superquantile as the risk measure and compare the resulting search plans with the ones obtained using the expectation. We are particularly interested in cases where the probability distribution of the parametrized random variable is unknown, but a table of realizations is available.

## **B. CONTRIBUTIONS**

This thesis is the first to consider superquantile as a risk measure in the context of search planning. Existing approaches focus exclusively on expectation. Since the superquantile is averse as explained below, it allows the analyst to plan for rare and undesirable events. We also consider search problems with incomplete information about sensor performance and we construct function approximations based on nonstandard regression models for use in subsequent optimization problems.

## **C. THESIS ORGANIZATION**

Chapter II defines the search problem we are interested in and formulates a model that will be used for theoretical and numerical studies. The chapter describes three distinct cases and scenarios defined by the sensor detection rates and target paths used in numerical studies.

Chapter III presents two of the cases where the probability distribution of the detection rate is known. We solve and compare these cases for expectation and superquantile risk measures.

Chapter IV discusses the third case, where the probability distribution of the detection rate is unknown and the decision maker only has a table of realizations of detection rates, collected for a certain time period in the past, for a certain visibility condition. We use a nonstandard regression model, and the obtained regression coefficients are then used in the search problem as an approximation.

Chapter V summarizes the theoretical and numerical results, and recommends further research.

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## II. MODEL FORMULATION AND RISK MEASURE

This chapter describes a search problem whose solutions we compare in the following chapters under various assumptions.

### A. PROBLEM DEFINITION

In this thesis, we study a search-optimization problem (SP) where we consider an Area Of Interest (AOI) defined by a finite set of cells  $\mathcal{C} = \{1, 2, \dots, C\}$ , and we let time also be discrete, described as a finite set of time periods  $\mathcal{T}_0 = \{0\} \cup \mathcal{T}$ , where  $\mathcal{T} = \{1, 2, \dots, T\}$ .

We want to route search assets in this AOI in order to detect a target such as a drug smuggler who is trying to cross it. The AOI includes a given number of bases, located in specific cells, with different logistical capabilities. Each one of these bases is able to accommodate a given number of search assets.

Based on Royset and Sato (2010), we consider a single target, but an arbitrary number of searchers. Both target and searchers have their own initial positions in the AOI. The initial positions for the searchers correspond to bases they are allocated to at time 0 or other cells in the AOI representing some area from which they start their missions, accounting for situations where some assets might have already been allocated for another low priority mission. We assume that the searchers divide their effort across multiple cells in arbitrarily small portions. This assumption allows for continuous variables in the model formulation. Many of the results in this thesis, however, generalize straightforwardly to the integer case.

At each time period  $t \in \mathcal{T}$ , searchers and target either occupy a cell  $c$  or are transiting between cells. When in cell  $c$ , they are allowed to move to any cell which is adjacent to  $c$  as defined by  $\mathcal{S}(c) \subseteq \mathcal{C}$ .

We assume that each searcher is equipped with one imperfect sensor, which

is not subject to false-positive detections. We also assume that the searchers are all alike, equipped with the same type of sensor. At each time period  $t \in \mathcal{T}$  during which a searcher is in cell  $c$ , its sensor takes one glimpse to see if the target is present in the cell. We define  $R_{c,t}$  as the corresponding nonnegative detection rate, in cell  $c$  and time period  $t$ , of the sensor. The increased noise caused by the searcher movements between cells affects the sensor performance. Therefore we assume that the sensor is inactive during transit between cells. Furthermore its performance is uncertain due to unknown environmental conditions and target characteristics.

In the AOI there is one target whose initial position is obtained from intelligence reports. The target moves conditionally deterministic and a possible target path is denoted by  $\omega = (\omega_1, \omega_2, \dots, \omega_T) \in \Omega \subset \mathcal{C}^T$ , where  $\omega_t \in \mathcal{C}$  is the target's cell at time period  $t$  on path  $\omega$ , with a corresponding probability  $\delta(\omega)$  of the target actually taking the path  $\omega$ , where  $\Omega$  is the set of all possible paths. We let  $P_{c,t}$  be a Bernoulli random variable, taking the value 1 if the target is in cell  $c$  at time period  $t$ . We define  $\gamma(v)$  as the probability of a particular scenario  $v$  occurring, where  $v$  is a realization of  $V = (V_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}} = ((P_{c,t}, R_{c,t})^\top)_{c \in \mathcal{C}, t \in \mathcal{T}}$ , a vector representing the random target path and random detection rates.

Our goal is to determine a plan that routes the available search assets over the AOI such that the probability of missing the target is minimized, without the full knowledge of future environmental conditions that might affect the sensor performance.

## B. MODEL FORMULATION

We formulate the search problem as a convex nonlinear program, similar to Royset and Sato (2010), but we generalize it by using  $\mathcal{R}(\cdot)$  to denote a risk measure as defined in the following section. The model SP takes the following form:



## Model SP:

### Indices

$c, c'$	cells ( $c, c' \in \mathcal{C} = \{1, \dots, C\}$ ).
$t$	time periods ( $t \in \mathcal{T}_0 = \{0\} \cup \mathcal{T}$ , $\mathcal{T} = \{1, \dots, T\}$ ).

### Set

$\mathcal{S}(c) \subseteq \mathcal{C}$	allowable search moves the searchers can carry out starting from cell $c$ .
--	---

### Parameters

$d_{c,c'}$	number of time periods needed for a searcher to move directly from cell $c$ to cell $c'$ and search $c'$ .
$x_{c,0}$	number of searchers positioned at cell $c$ in time period 0.

### Random Variables

$P_{c,t}$	1 if target is in cell $c$ in time period $t$ , otherwise 0; probability distribution of $P_{c,t}$ defined by $\delta(\omega)$ through $\text{Prob}(P_{c,t} = 1) = \text{Prob}(\omega_t = c)$ .
$R_{c,t}$	nonnegative detection rate in cell $c$ at time $t$ .
$V$	vector of random variables $V = (V_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}} = ((P_{c,t}, R_{c,t})^\top)_{c \in \mathcal{C}, t \in \mathcal{T}}$ , with realization $v = (v_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}} = ((p_{c,t}, r_{c,t})^\top)_{c \in \mathcal{C}, t \in \mathcal{T}}$ and $\text{Prob}(V = v) = \gamma(v)$ .

### Decision Variables

$x_{c,c',t}$  number of searchers that occupy cell  $c$  at time  $t$   
and move to cell  $c'$  next, with  $x = (x_{c,c',t})_{c,c' \in \mathcal{C}, t \in \mathcal{T}}$ .

### Function

$F(x, v)$  probability of missing the target given search plan  $x$  and realization  $v$  of  $V$ , where

$$F(x, v) = \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\}. \quad (\text{II.1})$$

### Formulation

$$\min_x \mathcal{R}(F(x, V)) \quad (\text{II.2})$$

$$\text{s.t.} \quad \sum_{c' \in \mathcal{S}(c)} x_{c',c,t-d_{c',c}} = \sum_{c' \in \mathcal{S}(c)} x_{c,c',t} \quad \forall c, t \in \mathcal{T} \quad (\text{II.3})$$

$$\sum_{c' \in \mathcal{S}(c)} x_{c,c',0} = x_{c,0} \quad \forall c \quad (\text{II.4})$$

$$x_{c,c',t} \geq 0 \quad \forall c, c', t \quad (\text{II.5})$$

We adopt the same function as in Royset and Sato (2010) for the probability of missing the target given a search plan  $x$  and a realization  $v$ , as seen in (II.1). The constraint (II.3) defines the allowable moves the searchers may take and the corresponding duration in time periods, avoiding jumps between non-adjacent cells, and the constraint (II.4) establishes the searcher initial positions. We discuss the numerical results of the model SP for different assumptions about the distribution of  $R_{c,t}$  and risk measures in Chapters III and IV.

## C. RISK MEASURE

One approach to SP is to choose a particular scenario  $v$ , and then minimize  $F(x, v)$  subject to (II.3)-(II.5). This means that the variability of the future is not taken into account and there is no guarantee that the obtained search plan  $x$  is optimal for  $F(x, v')$ , where  $v' \neq v$  is the observed scenario during the mission. The searchers might not be prepared for a scenario that deviates from the one chosen. Another alternative is to consider the worst-case scenario and minimize  $\sup_v F(x, v)$  subject to (II.3)-(II.5). The resulting solution may be highly conservative. Others prefer to rely on the expected value of the objective function, which corresponds to the probability of missing the target. On average they do well, but the resulting search plan might be quite poor in a given scenario. We compare different choices of this kind, which correspond to different risk measures  $\mathcal{R}(\cdot)$ , and see how well the resulting search plans perform in some sense.

As seen in Rockafellar et al. (2008), we say that a risk measure is coherent if the following axioms hold:

- (i)  $\mathcal{R}(C) = C$  for a constant  $C$ .
- (ii)  $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$  when  $\lambda > 0$  (positive homogeneity).
- (iii)  $\mathcal{R}(X + X') \leq \mathcal{R}(X) + \mathcal{R}(X')$  (subadditivity).
- (iv)  $\mathcal{R}(X) \leq \mathcal{R}(X')$  when  $X \leq X'$  (monotonicity).

We here assume that the random variable  $X$ , to which  $\mathcal{R}(\cdot)$  is applied is oriented such that large values are undesirable. In SP the random variable is  $F(x, V)$ . If  $\mathcal{R}(X) > E[X]$ , for a nonconstant random variable  $X$ , then  $\mathcal{R}(\cdot)$  is averse. Obviously, the expectation is not averse.

Based on Rockafellar and Uryasev (2000), we focus on the  $\alpha$ -superquantile risk measure

$$\mathcal{R}(X) = \bar{q}_\alpha(X), \tag{II.6}$$

which is coherent and averse for  $\alpha > 0$ , where  $\bar{q}_\alpha(X)$  is the  $\alpha$ -superquantile of  $X$  defined by

$$\bar{q}_\alpha(X) = E[X \mid X \geq q_\alpha(X)], \quad (\text{II.7})$$

with  $q_\alpha(X)$  being the  $\alpha$ -quantile of  $X$ , and  $\alpha \in [0, 1]$  being a probability level. We note that for a probability level of  $\alpha = 0$ , we obtain that  $E[X] = \bar{q}_0(X)$ , and for a probability level of  $\alpha = 1$ , we get  $\sup X = \bar{q}_1(X)$ , i.e., it is equivalent to analyzing the worst-case scenario. We let SP-E denote SP with  $\mathcal{R}(\cdot) = E[\cdot]$  and SP- $S_\alpha$  denote SP with  $\mathcal{R}(\cdot) = \bar{q}_\alpha(\cdot)$ .

By Rockafellar and Uryasev (2000),

$$\bar{q}_\alpha(F(x, V)) = \min_z z + \frac{1}{1 - \alpha} E[\max\{F(x, V) - z, 0\}]. \quad (\text{II.8})$$

Hence SP- $S_\alpha$  involves optimizing over  $x$  and the auxiliary variable  $z$ , and the objective function

$$z + \frac{1}{1 - \alpha} E[\max\{F(x, V) - z, 0\}]. \quad (\text{II.9})$$

SP- $S_\alpha$  can be reformulated as a large-scale convex smooth nonlinear program (Rockafellar & Uryasev, 2000). However, we implement SP- $S_\alpha$  with the exponential smoothing technique (Kort & Bertsekas, 1972) and obtain the following approximate objective function

$$z + \frac{1}{1 - \alpha} E[G_p(x, z, V)], \quad (\text{II.10})$$

where

$$G_p(x, z, v) = \frac{1}{p} \ln(\exp\{p(F(x, v) - z)\} + 1) \quad (\text{II.11})$$

and  $p > 0$ , which is a smoothing parameter.  $G_p(\cdot, \cdot, v)$  is a continuously differentiable function for all  $v$ . With some mathematical manipulations, we can see that

$$0 \leq G_p(x, z, v) - G(x, z, v) \leq \frac{\ln 2}{p} \quad (\text{II.12})$$

for all  $x, z, v$ , and  $p > 0$ , where

$$G(x, z, v) = \max\{F(x, v) - z, 0\}, \quad (\text{II.13})$$

see, e.g., Pee and Royset (2011).

## D. SCENARIO AND TARGET PATHS

We consider an AOI represented by a grid of eleven by eleven cells, numbered from right to left, from top to bottom, as shown in Figure 1.

11	10	9	8	7	6	5	4	3	2	1
22	21	20	19	18	17	16	15	14	13	12
33	32	31	30	29	28	27	26	25	24	23
44	43	42	41	40	39	38	37	36	35	34
55	54	53	52	51	50	49	48	47	46	45
66	65	64	63	62	61	60	59	58	57	56
77	76	75	74	73	72	71	70	69	68	67
88	87	86	85	84	83	82	81	80	79	78
99	98	97	96	95	94	93	92	91	90	89
110	109	108	107	106	105	104	103	102	101	100
121	120	119	118	117	116	115	114	113	112	111

Figure 1. AOI represented by a grid of 11 by 11 cells.

The searcher initial position is cell 61. According to information gathered via intelligence describing the target position in cell 66 ten time periods prior to the mission start, we define three distributions for the target initial location as shown in Figures 3, 4, and 5. We obtain these target initial distributions using a Markov chain target model for different stationary probabilities, denoted by  $\rho$ , which represent the probability of the target staying in the same cell for the following time period. Figure 2 represents the allowable movements denoted by  $\mathcal{S}(c)$ . The sum of the probabilities

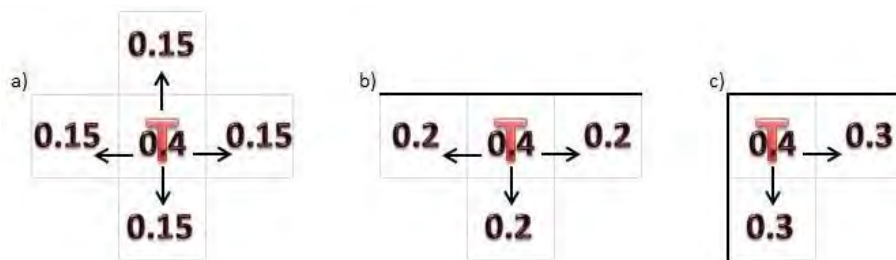


Figure 2. Example of target set of allowable movements for  $\rho = 0.4$ : a) in the middle of the AOI; b) at the boundary of the AOI; c) at the corner of the AOI.

of the target moving to the adjacent cells is  $1 - \rho$ , and all these allowable cells have equally likely probabilities.

We consider three different target stationary probabilities  $\rho$  and obtain three corresponding initial target distributions, the first for  $\rho = 0.6$ , the second for  $\rho = 0.4$ , and the last for  $\rho = 0.2$ ; see Figures 3, 4, and 5, respectively. The numbers in the

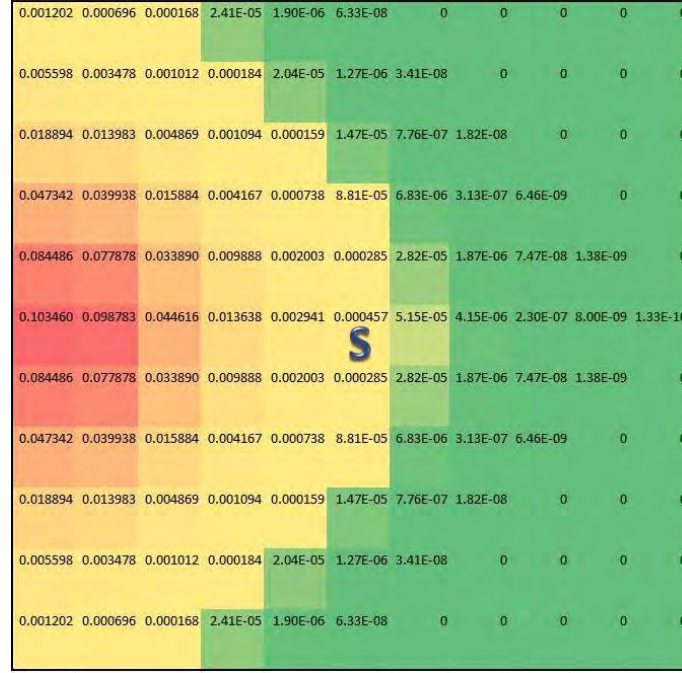


Figure 3. Searcher (S) initial position and target initial distribution for  $\rho = 0.6$ .

cells on Figures 3, 4, and 5, correspond to the probability of the target being in that particular cell at time period 1, given the target stationary probability  $\rho$ . The more red the cell is, the higher the probability of the target being in that cell. A larger  $\rho$  implies a faster target.

For this thesis, we randomly generate 100,000 independent target paths from the Markov chain induced by  $\rho$  and  $\mathcal{S}(c)$ , for the three distinct values of  $\rho$ . We here use a Markov chain for simplicity in implementation. Any target motion model can be used. In reality these paths could arrive via intelligence prior to the mission start,

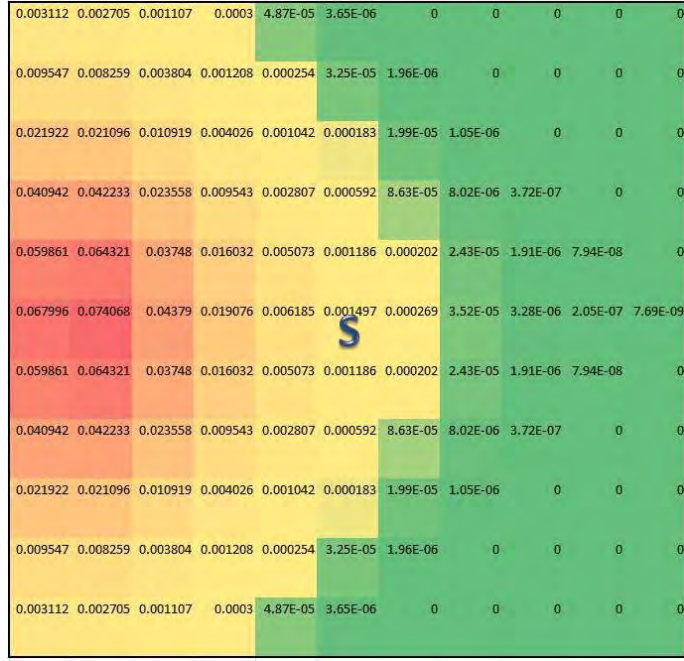


Figure 4. Searcher (S) initial position and target initial distribution for  $\rho = 0.4$ .

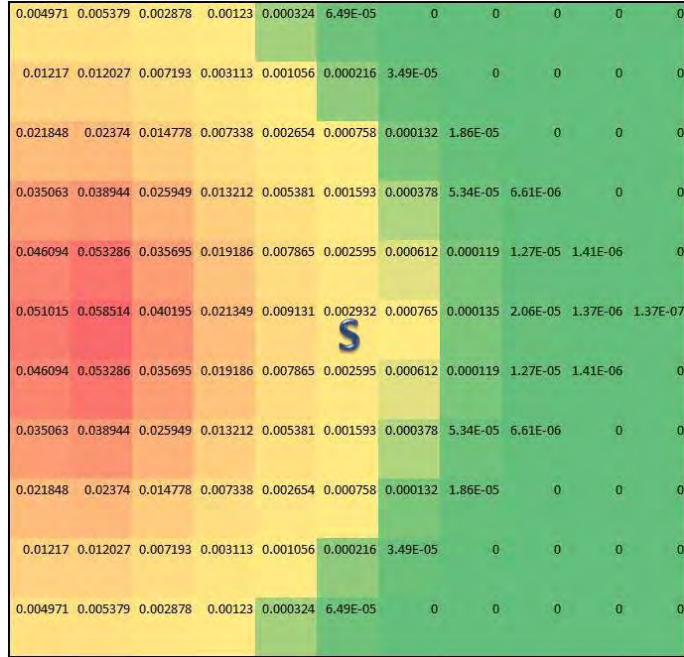


Figure 5. Searcher (S) initial position and target initial distribution for  $\rho = 0.2$ .

e.g., data acquired by AIS (Automatic Identification System), a tracking system used on board ships.

## E. CASES

In the next two chapters, we consider three different cases of SP. In Case A we assume that the detection rates  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  are deterministic, where in Case B the detection rates are random with a known probability distribution. We discuss Cases A and B in detail in Chapter III. In Case C, the probability distribution of the detection rate is unknown. We use regression techniques to approximate the detection rate by a linear combination of well-known factors. We consider Case C in Chapter IV.



### III. SEARCH PLANS UNDER DETERMINISTIC AND RANDOM DETECTION RATE

This chapter considers Cases A and B, as described in Section II.E. We use the expectation and the superquantile as the risk measures of the random objective function, and we compare their corresponding results.

#### A. DETERMINISTIC DETECTION RATE

In this section, we are interested in situations where the random detection rate  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  takes on deterministic values. We use the expectation and superquantile, models SP-E and SP-S $_{\alpha}$ , respectively, as explained in Section II.C. We first consider the situation where the detection rate is constant over all cells and time periods.

Using the information contained in USCG (2009) as a reference, and for illustration purposes we define visibility conditions as poor, fair, or good and the corresponding visual ranges in nautical miles (nm) are given in Table 1. In poor, fair,

Visibility	Visual Range (nm)	Deterministic Detection Rate $r_{c,t}$
Poor	1	1.1824
Fair	5	1.8237
Good	10	2.8715

Table 1. Deterministic detection rate  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  for poor, fair and good visibility.

and good visibility, we set the probability of detecting a target in a cell during one time period, given that the target and a searcher are present, to values that when used in the numerical examples return optimal objective function values. We refer to this probability as the glimpse detection probability  $g_{c,t}$ . The detection rate  $r_{c,t}$  then follows using the relationship  $r_{c,t} = -\ln(1 - g_{c,t})$ ; see Royset and Sato (2010).

Table 1 gives the resulting detection rates. These chosen detection rates allow us to compare results from this chapter with results obtained in Chapter IV, but are not representative of real search sensor performances.

We are also interested in situations where the detection rate  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  is deterministic, but varies between cells and time periods. So we construct three problem instances with such detection rates by independently sampling from three lognormal distributions with parameters  $\mu = \ln(m_i) - \sigma^2/2$ , where  $m_i$ ,  $i = 1, 2, 3$ , are the detection rates of Table 1 and  $\sigma = 0.1$  such that the lognormal distributions have means corresponding to those in Table 1. We independently generate one detection rate per cell and time period and we refer to Table 2 for the sample statistics summary.

Visibility		Sample Mean	Sample Variance
Poor		1.1846	0.0138
Fair		1.8259	0.0335
Good		2.8784	0.0829

Table 2. Summary of the sampled detection rates for the given visibility conditions.

This results in 1,210 sampled detection rates, with smoothed density plots given in Figure 6.

## B. RANDOM DETECTION RATE WITH KNOWN DISTRIBUTION

We now discuss Case B where the detection rate  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  is random with a known probability distribution. The detection rate is constant for all cells and time periods, but that constant is random.

For Case B, we also consider the same three visibility conditions, and we construct three random variables with lognormal distributions having the same parameters as for Case A, describing the detection rate for the three different visibility conditions, in order to compare the results of Cases A and B. Here we generate 10

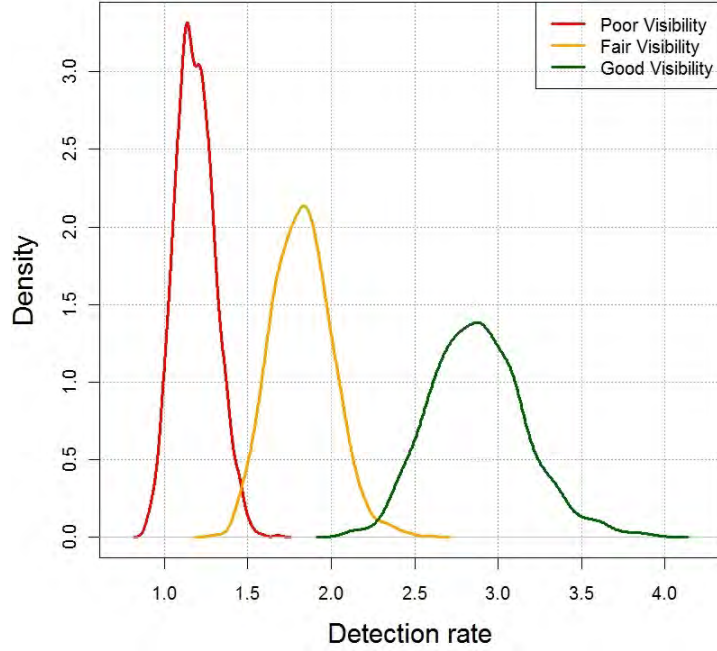


Figure 6. Sampled detection rate density plots for poor, fair, and good visibility.

lognormal detection rates per visibility condition; see Table 3 for the sample statistics summary.

Visibility		Sample Mean	Sample Variance
Poor		1.2261	0.0128
Fair		1.7281	0.0334
Good		2.9425	0.1267

Table 3. Summary of the 10 sampled detection rates for poor, fair, and good visibility.

During the next section we discuss and analyze the results obtained for Cases A and B, considering different risk measures and probability levels.

### C. NUMERICAL RESULTS

We implement SP in the programming language General Algebraic Modeling System (GAMS) on a personal computer with 8.00 GB of RAM and 2.80 GHz

processor running Windows 7. SP is solved using the MINOS solver with default options.

We let  $T = 10$ , and fix  $p = 500$  for the exponential smoothing technique described in Section II.C. The corresponding smoothing error is bounded by 0.001386, which is insignificant in our context.

## 1. Case A - Deterministic Detection Rate

Table 4 shows the obtained results for Case A where we have a constant detection rate equal for all cells and time periods, for various target stationary probabilities  $\rho$ , visibility conditions, and probability levels  $\alpha$ . Although we do not show the results for  $\alpha = 0$ , the corresponding probabilities of missing the target are exactly the same as the ones obtained using SP-E.

$\rho$	Visibility	SP-E	SP-S $_{\alpha}$			
			$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.90$
0.6	Poor	0.6517	0.7399	0.8016	0.8577	0.8912
0.6	Fair	0.5492	0.6402	0.7161	0.7903	0.8369
0.6	Good	0.4303	0.5171	0.6018	0.6939	0.7564
0.4	Poor	0.7212	0.8059	0.8557	0.8940	0.9142
0.4	Fair	0.6302	0.7241	0.7884	0.8413	0.8699
0.4	Good	0.5197	0.6152	0.6932	0.7634	0.8032
0.2	Poor	0.7619	0.8406	0.8813	0.9101	0.9245
0.2	Fair	0.6788	0.7693	0.8238	0.8646	0.8848
0.2	Good	0.5742	0.6710	0.7401	0.7962	0.8244

Table 4. Case A - Optimal objective function values for models SP-E and SP-S $_{\alpha}$ , with  $\alpha = 0.25, 0.50, 0.75$  and  $0.90$ , for a constant detection rate equal for all cells and time periods, where the objective function is the risk measure of the probability of missing the target.

From Table 4 we observe that the better the visibility conditions the smaller the probabilities of missing the target, as we would expect. For larger values of  $\rho$ ,

we obtain the same result. Thus for better visibility and a slower moving target, the searchers have a better chance of detecting the target. We also observe that the difference between the objective function values for different probability levels  $\alpha$  decreases with the stationary probability  $\rho$ , e.g., for poor visibility, the difference between the results obtained for  $\alpha = 0.75$ , and  $\alpha = 0.90$  is smaller for  $\rho = 0.2$  than for  $\rho = 0.6$  ( $0.8912 - 0.8577 > 0.9245 - 0.9101$ ).

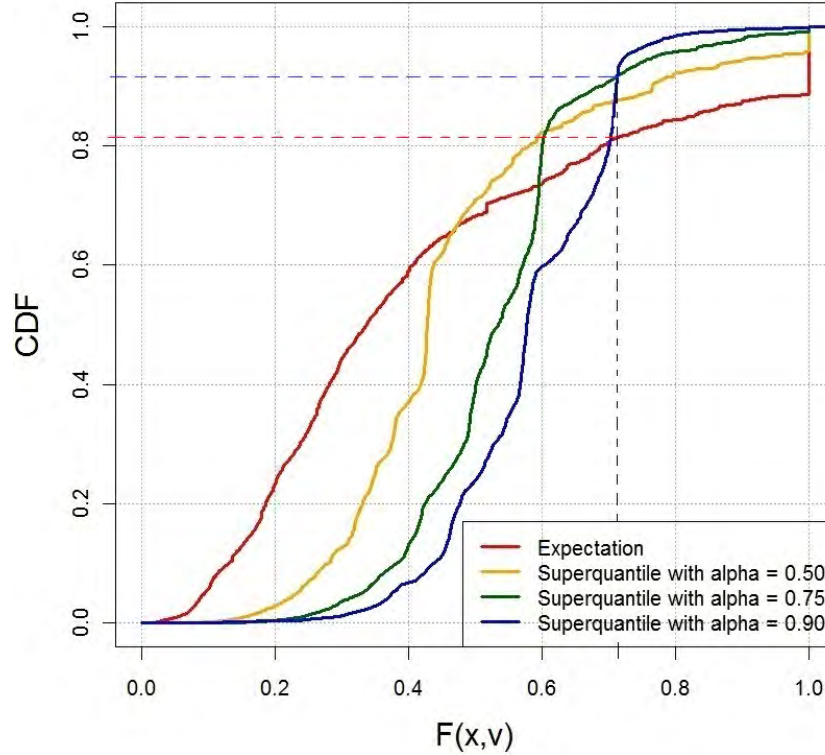


Figure 7. Case A - Cumulative distribution functions for the probability of missing the target given optimal search plan  $x$  and deterministic detection rate (good visibility,  $\rho = 0.6$ ).

In Figures 7, 8, and 9, we compare the cumulative distribution functions for  $F(x, V)$  for the corresponding optimal search plans, when the visibility condition is good, with  $\rho = 0.6$ ,  $\rho = 0.4$ , and  $\rho = 0.2$ , respectively. There is a notable distinction between using expectation and the superquantile with different probability levels  $\alpha$ . If the decision maker wants to make sure that the probability of missing the target does not exceed a certain threshold, then the best approach is to rely on the outcome

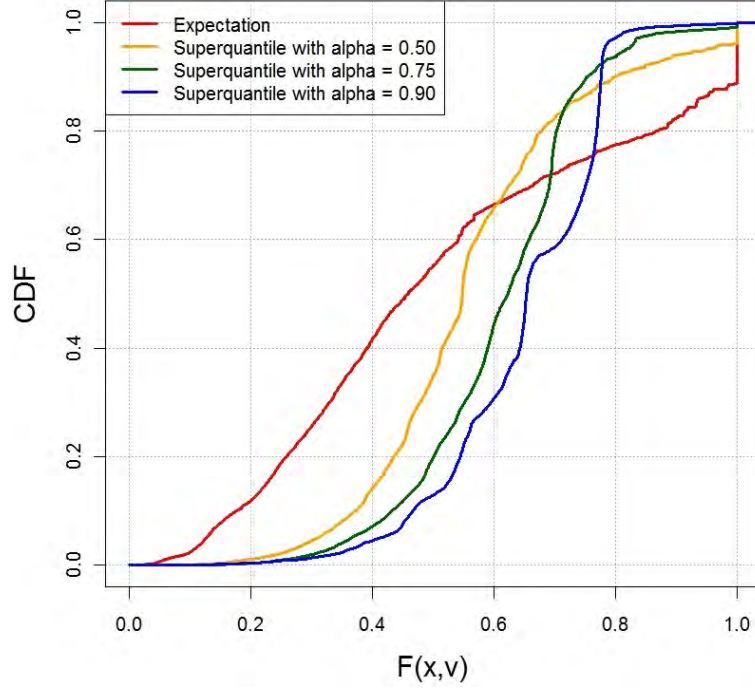


Figure 8. Case A - Cumulative distribution functions for the probability of missing the target given optimal search plan  $x$  and deterministic detection rate (good visibility,  $\rho = 0.4$ ).

of the model  $SP-S_\alpha$  for higher probability levels  $\alpha$ . Using Figure 7 as an example, if the goal is to prevent getting probabilities of missing the target of 70% or higher, then the decision maker should rely on the superquantile with probability level  $\alpha = 0.90$ , among the four risk measures that we present. With that choice there is a 92% chance of obtaining better mission outcomes than the goal, as indicated by the blue horizontal dashed line in Figure 7, while there is only an 81% chance of doing better than the goal when using the expectation; see the red dashed line. This difference might be critical when choosing a search plan. As we see in Figures 8 and 9, this difference is even more significant for smaller values of the stationary probability  $\rho$ , i.e., the faster the target moves, the more beneficial in some sense it is to use the model  $SP-S_\alpha$  with higher probability levels  $\alpha$ .

We also consider the situation where the detection rates vary between cells and time periods, and we obtain the results of Table 5, for distinct target stationary

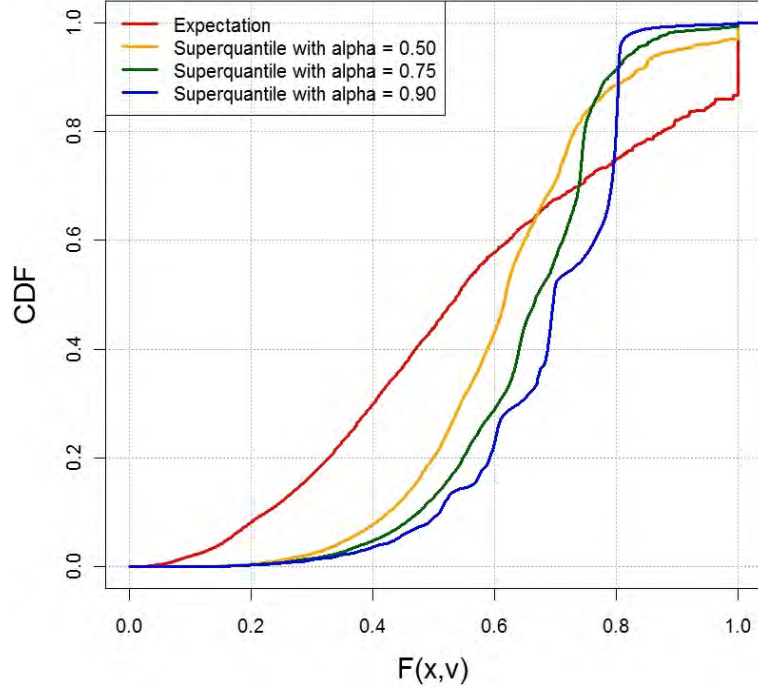


Figure 9. Case A - Cumulative distribution functions for the probability of missing the target given optimal search plan  $x$  and deterministic detection rate (good visibility,  $\rho = 0.2$ ).

probabilities  $\rho$ , visibility conditions, and probability levels  $\alpha$ .

The fact that almost all the obtained results in Table 5 are smaller than the ones from Table 4 is due to the fact that the detection rate changes over the AOI and time and allows the searcher to choose cells that for a given time period have better detection rates. Therefore we obtain smaller objective function values. With this example, we note the importance of good intelligence on visibility conditions since we obtain more accurate detection rates and consequently more precise objective function values.

Besides the probabilities of missing the target, it is useful to compare the corresponding search plans. Figures 10 and 11 illustrate the optimal search plans in Case A for good visibility conditions and target stationary probability  $\rho = 0.6$ , using SP-E and SP-S<sub>0.75</sub>, respectively. The horizontal axes show the cells in the AOI displayed by number and the time periods considered during the mission. The vertical

			SP-S $_{\alpha}$			
$\rho$	Visibility	SP-E	$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.90$
0.6	Poor	0.6451	0.7346	0.8004	0.8563	0.8904
0.6	Fair	0.5427	0.6363	0.7149	0.7913	0.8385
0.6	Good	0.4226	0.5122	0.6013	0.6947	0.7584
0.4	Poor	0.7159	0.8037	0.8543	0.8932	0.9135
0.4	Fair	0.6239	0.7214	0.7894	0.8430	0.8715
0.4	Good	0.5124	0.6127	0.6928	0.7646	0.8054
0.2	Poor	0.7572	0.8387	0.8803	0.9095	0.9242
0.2	Fair	0.6734	0.7686	0.8256	0.8661	0.8860
0.2	Good	0.5679	0.6687	0.7401	0.7974	0.8263

Table 5. Case A - Optimal objective function values for models SP-E and SP-S $_{\alpha}$ , with  $\alpha = 0.25, 0.50, 0.75$  and  $0.90$ , for a detection rate varying between cells and time periods, where the objective function is the risk measure of the probability of missing the target.

axis is the fraction of searchers in cell  $c$  at time period  $t$ . The color red turning into black represents the time evolution.

Comparing Figures 10 and 11, we realize that the model SP-S $_{0.75}$  spreads the searchers over the AOI, covering a larger area with smaller searcher fractions. Hence, the plan more effectively handles exceptional target paths.

## 2. Case B - Random Detection Rate With Known Distribution

Table 6 shows the obtained results for Case B where we have a constant detection rate equal for all cells and time periods, whose value is random, for various target stationary probabilities  $\rho$ , and probability levels  $\alpha$ , when we have good visibility. Since the numerical results are similar to the ones of Case A, we do not present the results for poor and fair visibility. From Table 6 we observe that the obtained optimal objective function values increase with the value of  $\alpha$ , similar to Case A.



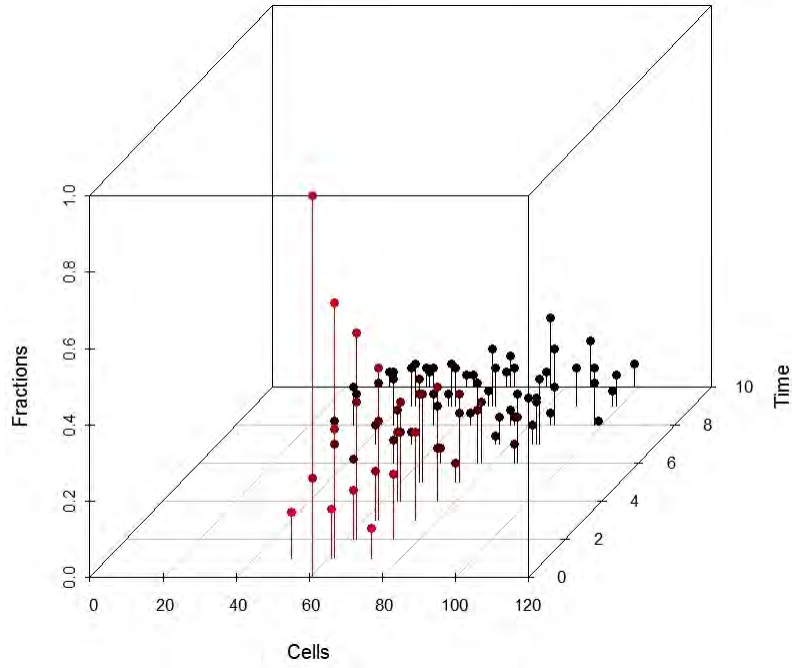


Figure 10. Case A - Optimal search plan for the model SP-E, with  $\rho = 0.6$ , for a detection rate constant for all cells and time periods.

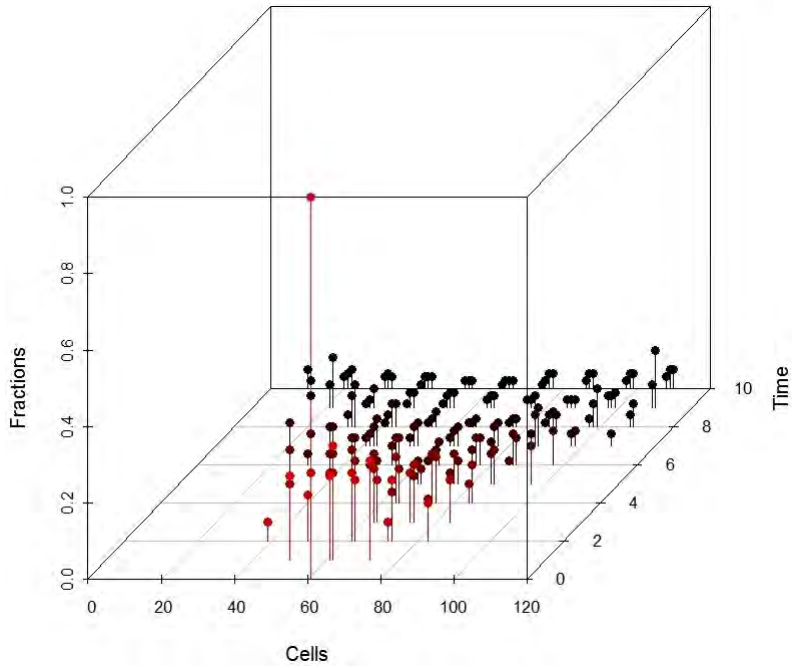


Figure 11. Case A - Optimal search plan for the model SP -  $S_\alpha = 0.75$ , with  $\rho = 0.6$ , for a detection rate constant for all cells and time periods.

$\rho$	Visibility	SP-E	SP-S $_{\alpha}$			
			$\alpha = 0.25$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.90$
0.6	Good	0.4142	0.5027	0.5906	0.6887	0.7602
0.4		0.5041	0.6013	0.6830	0.7539	0.8131
0.2		0.5593	0.6579	0.7309	0.7949	0.8399

Table 6. Case B - Optimal objective function values for models SP-E and SP-S $_{\alpha}$ , with  $\alpha = 0.25, 0.50, 0.75$  and  $0.90$ , for a detection rate constant over all cells and time periods, where the objective function is the risk measure of the probability of missing the target.

In Figure 12 we compare the cumulative distribution functions of  $F(x, V)$  for the corresponding optimal search plans for good visibility conditions, with  $\rho = 0.4$ . We notice a clear distinction between the four risk measures. In this example, the goal

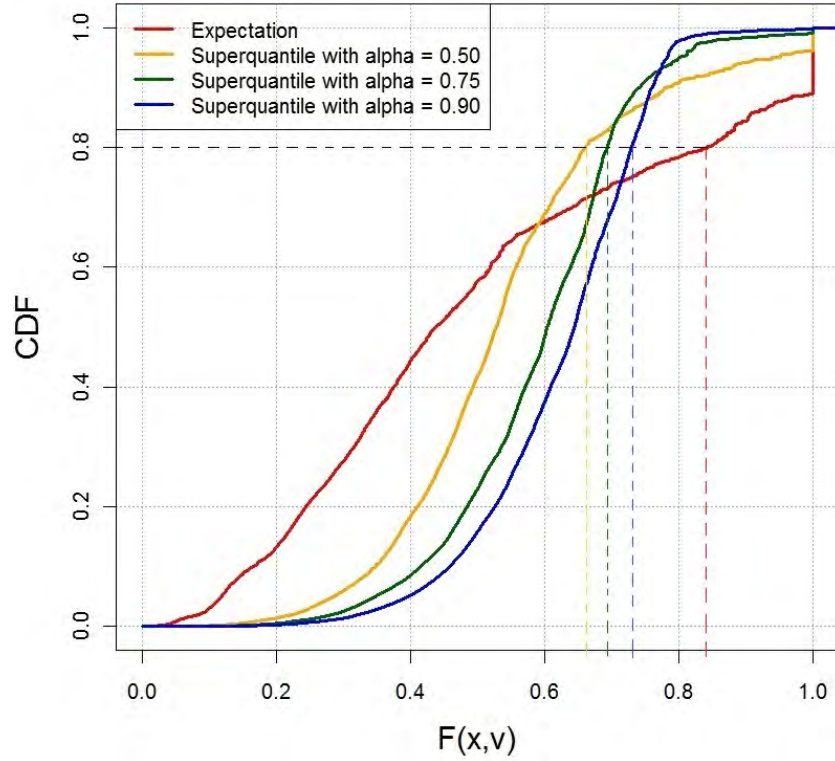


Figure 12. Case B - Cumulative distribution functions for the probability of missing the target given optimal search plan  $x$  and random detection rate constant over all cells and time periods (good visibility,  $\rho = 0.4$ ).

is to establish a threshold for  $F(x, V)$  and consider the worst 20% of scenarios. In Figure 12, a horizontal dashed line marks the 80% of scenarios. The vertical dashed lines mark the corresponding thresholds for the four risk measures. The horizontal dashed line crosses each risk measure cumulative distribution function and gives corresponding probabilities of missing the target on the horizontal axis, as indicated by the colored vertical dashed line. In this particular example, we note that the worst 20% of scenarios involve a probability of missing the target greater than 84% for the expectation, where as only 65% for the superquantile with  $\alpha = 0.50$ . Hence, the decision maker should rely on the superquantile with  $\alpha = 0.50$  and therefore avoid higher probabilities of missing the target.

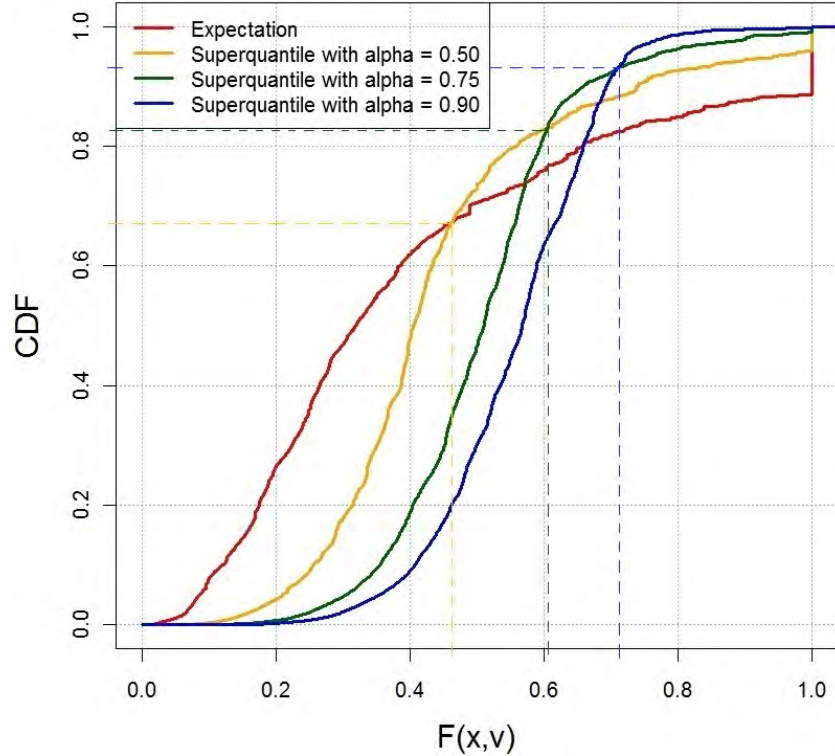


Figure 13. Case B - Cumulative distribution functions for the probability of missing the target given optimal search plan  $x$  and random detection rate constant over all cells and time periods (good visibility,  $\rho = 0.6$ ).

Figure 13 shows a final example for Case B. In this situation we assume that the decision maker is concerned with what risk measure to use in the objective function

for a given threshold. Once the threshold is established, we note which cumulative distribution function has larger value. Here we check which interval between vertical dashed lines the threshold is in, and the color of the smallest of the two lines indicates which risk measure the decision maker should rely on. For example, if the threshold is 65%, then the decision maker should rely on the superquantile with  $\alpha = 0.75$ , as shown by the green dashed line. Similarly, if the threshold is 80%, then the probability level should be  $\alpha = 0.90$ . We note that we should rely on expectation only if the threshold is smaller than 46% for the probability of missing the target.

## IV. SEARCH PLANS UNDER UNKNOWN DETECTION RATE

This chapter considers Case C, as described in Section II.E. We assume that the probability distribution of the detection rate is unknown, however the decision maker has a table of realizations available.

The detection rate performance of the sensor equipped on board a searcher is usually complicated to determine. We assume that there exist some factors that describe the detection rate in some sense and whose probability distributions we know and are easy to handle. These factors could be environmental conditions and target characteristics. We next discuss how to approximate the detection rates using these factors.

### A. APPROXIMATION OF RANDOM DETECTION RATE

Suppose that we adopt the approximation

$$R_{c,t} \approx \hat{R}_{c,t} = \hat{R}_{c,t}(\beta_{c,t}^0, \beta_{c,t}^\top) = \beta_{c,t}^0 + \beta_{c,t}^\top Y, \quad c \in \mathcal{C}, t \in \mathcal{T}, \quad (\text{IV.1})$$

where  $Y$  is a vector of factors with known joint distribution and  $\beta_{c,t}^0$  and  $\beta_{c,t}$  are regression coefficients. Then the approximate probability of missing the target becomes

$$F(x, V) \approx F(x, \hat{V}) = \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} (\beta_{c,t}^0 + \beta_{c,t}^\top Y) x_{c, c', t} \right\}. \quad (\text{IV.2})$$

Since we approximate the random variable  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  by a linear combination of different factors, we need to ensure that the difference between the risk measure of the true random objective function and the risk measure of the approximated one,  $\mathcal{R}(F(x, V)) - \mathcal{R}(F(x, \hat{V}))$ , is bounded in some sense, giving us a reasonable approximation to use in SP. The following theorem proves such bounds.

**Theorem IV.1.** *For a risk measure  $\mathcal{R}(\cdot)$  that satisfies the axioms stated in section II.C, if we have  $\hat{R}_{c,t} \geq 0$  almost surely for all  $c \in \mathcal{C}$  and  $t \in \mathcal{T}$ , then for any  $x \geq 0$*

$$\begin{aligned}
& - \sum_{c,c',t \in \mathcal{T}} x_{c,c',t} \mathcal{R} \left( \max \left\{ 0, R_{c,t} - \hat{R}_{c,t} \right\} \right) \\
& \leq \mathcal{R} (F(x, V)) - \mathcal{R} (F(x, \hat{V})) \\
& \leq \sum_{c,c',t \in \mathcal{T}} x_{c,c',t} \mathcal{R} \left( \max \left\{ 0, \hat{R}_{c,t} - R_{c,t} \right\} \right). \tag{IV.3}
\end{aligned}$$

**Proof.**

For any  $x \geq 0$  and realizations  $v = ((p_{c,t}, r_{c,t})^\top)_{c \in \mathcal{C}, t \in \mathcal{T}}$  and  $\hat{v} = ((p_{c,t}, \hat{r}_{c,t})^\top)_{c \in \mathcal{C}, t \in \mathcal{T}}$  of  $V$  and  $\hat{V}$ , respectively, we have that

$$F(x, v) - F(x, \hat{v}) = \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\} - \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right\}. \tag{IV.4}$$

Thus,

$$|F(x, v) - F(x, \hat{v})| \leq \left| \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} - \sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right|, \tag{IV.5}$$

where we use the fact that the exponential function is Lipschitz continuous with constant 1, provided that we are dealing almost surely with non-positive exponents. Since  $\hat{r}_{c,t} \geq 0$  and  $p_{c,t} \geq 0$ , for the cases where  $\sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \geq \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t}$ , we obtain that

$$\begin{aligned}
F(x, v) - F(x, \hat{v}) &= \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\} - \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right\} \\
&\leq \sum_{c,c',t \in \mathcal{T}} p_{c,t} x_{c,c',t} \hat{r}_{c,t} - \sum_{c,c',t \in \mathcal{T}} p_{c,t} x_{c,c',t} r_{c,t}. \tag{IV.6}
\end{aligned}$$

On the other hand, for the cases where  $\sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \leq \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t}$ , we get

$$F(x, v) - F(x, \hat{v}) = \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\} - \exp \left\{ - \sum_{c,c',t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right\} \leq 0. \tag{IV.7}$$

Hence, for any  $\hat{r}_{c,t} \geq 0$ , combining both inequalities (IV.6) and (IV.7), and since  $p_{c,t} \geq 0$

$$\begin{aligned}
F(x, v) - F(x, \hat{v}) &= \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c, c', t} \right\} - \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c, c', t} \right\} \\
&\leq \max \left\{ 0, \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c, c', t} \hat{r}_{c,t} - \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c, c', t} r_{c,t} \right\} \\
&= \max \left\{ 0, \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c, c', t} (\hat{r}_{c,t} - r_{c,t}) \right\} \\
&\leq \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c, c', t} \max \{0, \hat{r}_{c,t} - r_{c,t}\} \\
&\leq \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \max \{0, \hat{r}_{c,t} - r_{c,t}\}. \tag{IV.8}
\end{aligned}$$

We therefore obtain that

$$F(x, v) \leq F(x, \hat{v}) + \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \max \{0, \hat{r}_{c,t} - r_{c,t}\}. \tag{IV.9}$$

The result holds if we take the risk measures on both sides, based on the axioms stated in section II.C,

$$\begin{aligned}
\mathcal{R}(F(x, V)) &\leq \mathcal{R} \left( F(x, \hat{V}) + \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \max \{0, \hat{R}_{c,t} - R_{c,t}\} \right) \\
&\leq \mathcal{R}(F(x, \hat{V})) + \mathcal{R} \left( \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \max \{0, \hat{R}_{c,t} - R_{c,t}\} \right) \\
&\leq \mathcal{R}(F(x, \hat{V})) + \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \mathcal{R} \left( \max \{0, \hat{R}_{c,t} - R_{c,t}\} \right). \tag{IV.10}
\end{aligned}$$

And we obtain the upper bound

$$\mathcal{R}(F(x, V)) - \mathcal{R}(F(x, \hat{V})) \leq \sum_{c, c', t \in \mathcal{T}} x_{c, c', t} \mathcal{R} \left( \max \{0, \hat{R}_{c,t} - R_{c,t}\} \right). \tag{IV.11}$$

In order to obtain a lower bound, we use the same approach as for the upper bound. Now for the cases where  $\sum_{c, c', t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c, c', t} \geq \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c, c', t}$ , we

obtain the inequality

$$F(x, \hat{v}) - F(x, v) = \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right\} - \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\} \leq 0. \quad (\text{IV.12})$$

And whenever  $\sum_{c, c', t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \leq \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t}$ , we get

$$\begin{aligned} F(x, \hat{v}) - F(x, v) &= \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} \hat{r}_{c,t} x_{c,c',t} \right\} - \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\} \\ &\leq \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c,c',t} r_{c,t} - \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c,c',t} \hat{r}_{c,t}. \end{aligned} \quad (\text{IV.13})$$

Hence, for any  $\hat{r}_{c,t} \geq 0$ , combining both inequalities (IV.12) and (IV.13), and since  $p_{c,t} \geq 0$

$$\begin{aligned} F(x, \hat{v}) - F(x, v) &\leq \max \left\{ 0, \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c,c',t} r_{c,t} - \sum_{c, c', t \in \mathcal{T}} p_{c,t} x_{c,c',t} \hat{r}_{c,t} \right\} \\ &\leq \sum_{c, c', t \in \mathcal{T}} x_{c,c',t} \max \{ 0, r_{c,t} - \hat{r}_{c,t} \}. \end{aligned} \quad (\text{IV.14})$$

After applying the same axioms as for the upper bound calculation, in (IV.10), we get the following lower bound

$$\mathcal{R}(F(x, V)) - \mathcal{R}(F(x, \hat{V})) \geq - \sum_{c, c', t \in \mathcal{T}} x_{c,c',t} \mathcal{R} \left( \max \{ 0, R_{c,t} - \hat{R}_{c,t} \} \right). \quad (\text{IV.15})$$

So combining both results stated before, we obtain the result.  $\square$

We use the result of Theorem IV.1 in the next section.

## B. RISK-TUNED REGRESSION MODELS

We would like to compute the regression coefficients  $\bar{\beta} = (\beta_{c,t}^0, \beta_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}^\top$  of (IV.1) in such a way that we minimize the difference  $\mathcal{R}(F(x, V)) - \mathcal{R}(F(x, \hat{V}))$ . In view of Theorem IV.1, we minimize a weighted sum of the lower and upper bounds as a surrogate of that difference. The weights  $w_1, w_2 \in [0, 1]$  are determined by the decision maker in a suitable manner. As an example on how to choose these weights, we have



that large probabilities may be more unfavorable in SP, therefore the upper bound should be more weighted. Overestimating the detection rate gives larger objective function values in SP, but in reality it translates into higher probabilities of missing the target.

Consequently we obtain the following optimization problem for determining the regression coefficients:

**Model RP( $w_1, w_2$ ):**

$$\min_{\bar{\beta}} \left\{ w_1 \sum_{c, c', t \in \mathcal{T}} \mathcal{R} \left( \max \left\{ 0, R_{c,t} - \hat{R}_{c,t}(\bar{\beta}) \right\} \right) + w_2 \sum_{c, c', t \in \mathcal{T}} \mathcal{R} \left( \max \left\{ 0, \hat{R}_{c,t}(\bar{\beta}) - R_{c,t} \right\} \right) \right\} \quad (\text{IV.16})$$

$$\text{s.t.} \quad \beta_{c,t}^0 + \beta_{c,t}^\top Y \geq 0 \quad \forall c, t \in \mathcal{T}, \text{ a.s.} \quad (\text{IV.17})$$

Constraint (IV.17) establishes that the approximated detection rate must be nonnegative, almost surely, for every cell and time period, since the true detection rate never takes on negative values. We note that RP( $w_1, w_2$ ) is a nonstandard regression problem.

We assume that we have a table of  $N$  realizations of the detection rates and the corresponding realizations of the factors. Let  $\{y^k, r^k\}_{k=1}^N$  be these realizations, where  $y^k$  and  $r^k$  correspond to the  $k^{th}$  realization of the factors  $Y$  and the detection rate  $R$ , respectively. For simplicity, we here assume that  $R_{c,t}$  is identical in all cells and all time periods. Our methodology applies also beyond this assumption. However, it then would require more data and calculations. In situations where the table of realizations depends on cell  $c$  and time period  $t$ , we have a total of  $CT$  auxiliary linear optimization programs to solve.

We use superquantile as the risk measure in RP( $w_1, w_2$ ), and with the discrete data mentioned above, we transcribe RP( $w_1, w_2$ ) into a linear program and compute the regression coefficients  $\bar{\beta} = (\beta^0, \beta)^\top$ , where we drop the subscripts  $c$  and  $t$  since all

cells and time periods are identical. We then use the obtained regression coefficients in the objective function of SP by replacing the true random vector  $(R_{c,t})_{c \in \mathcal{C}, t \in \mathcal{T}}$  by the approximated detection rate  $\hat{R}_{c,t} = \hat{R}_{c,t}(\beta^0, \beta^\top)$ , as seen in the following section.

### C. SEARCH PROBLEM UNDER UNKNOWN DETECTION RATE

In SP, after obtaining the regression coefficients, we substitute the true objective function

$$F(x, v) = \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} r_{c,t} x_{c,c',t} \right\}$$

by the following approximation

$$F(x, \hat{v}) = \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} (\beta^0 + \beta^\top y) x_{c,c',t} \right\}, \quad (\text{IV.18})$$

and we obtain a new model of SP as follows:

**Model  $\widehat{\text{SP}}$ :**

$$\min_x \mathcal{R} \left( \exp \left\{ - \sum_{c, c', t \in \mathcal{T}} p_{c,t} (\beta^0 + \beta^\top y) x_{c,c',t} \right\} \right) \quad (\text{IV.19})$$

$$\text{s.t.} \quad (\text{II.3}) - (\text{II.5})$$

In  $\widehat{\text{SP}}$  we use superquantile with a given probability level  $\alpha$  as the risk measure. The value of  $\alpha$  is the same one used in  $\text{RP}(w_1, w_2)$ .

In order to further simplify the computational effort for the next section, we only take into account one factor: visibility conditions in the AOI, measured in nautical miles, during the mission's time horizon. We consider situations where we might not have predictions of the visibility conditions for the mission time horizon, for example situations where the mission needs to be planned ahead in time and we do not know which visibility conditions to expect. We utilize the  $N$  realizations  $\{y^k, r^k\}_{k=1}^N$  in  $\text{RP}(w_1, w_2)$ . Then we use the distinct realizations  $\{y^k\}_{k=1}^N$  in  $\widehat{\text{SP}}$  to estimate a

probability distribution of  $Y$ , which we assume is independent of the probability distribution over the target paths, and then apply these distributions in  $\widehat{\text{SP}}$ .

In our numerical example, we assume that we have  $N = 30$  realizations, where  $y^1 = \dots = y^{10} = 1$ , and  $r^1, \dots, r^{10}$  are randomly generated using a lognormal distribution,  $y^{11} = \dots = y^{20} = 5$ , and  $r^{11}, \dots, r^{20}$  are randomly generated using a lognormal distribution, and  $y^{21} = \dots = y^{30} = 10$  nautical miles, and  $r^{21}, \dots, r^{30}$  are randomly generated using a lognormal distribution. All the lognormal distributions have the same parameters as for Case A, according to the visibility conditions. After obtaining the regression coefficients from the model  $\text{RP}(w_1, w_2)$ , we use the three distinct  $y^k$  in the model  $\widehat{\text{SP}}$ .

## D. NUMERICAL RESULTS

We next present numerical results of  $\text{RP}(w_1, w_2)$  and we compare the regression coefficients obtained using weights  $w_1 = 0.25$  and  $w_2 = 0.75$ , for different probability levels  $\alpha$ , with the traditional least-squares linear regression. We show two different plots, one to represent  $\text{RP}(0.25, 0.75)$ , the other to represent  $\text{RP}(0.75, 0.25)$ , in order to demonstrate that the values of the weights  $w_1$  and  $w_2$  are fairly important.

$\alpha$		Axis Interception $\beta_0$		Slope $\beta$	
0		0.9567		0.1645	
0.25		0.9567		0.1645	
0.35		0.9367		0.1665	
0.50		0.8740		0.1728	
0.75		0.8024		0.1750	
0.80		0.6378		0.1816	
0.85		0.6925		0.1707	
0.90		0.6967		0.1698	

Table 7. Case C - Regression coefficients  $\bar{\beta} = (\beta^0, \beta)^\top$  for different probability levels  $\alpha$ , in  $\text{RP}(0.25, 0.75)$ .

Table 7 shows the obtained regression coefficients  $\bar{\beta} = (\beta^0, \beta)^\top$  for different probability levels  $\alpha$ , and weights  $w_1 = 0.25, w_2 = 0.75$ . In the traditional least-squares regression we obtain  $\bar{\beta} = (0.9665, 0.1886)^\top$ .

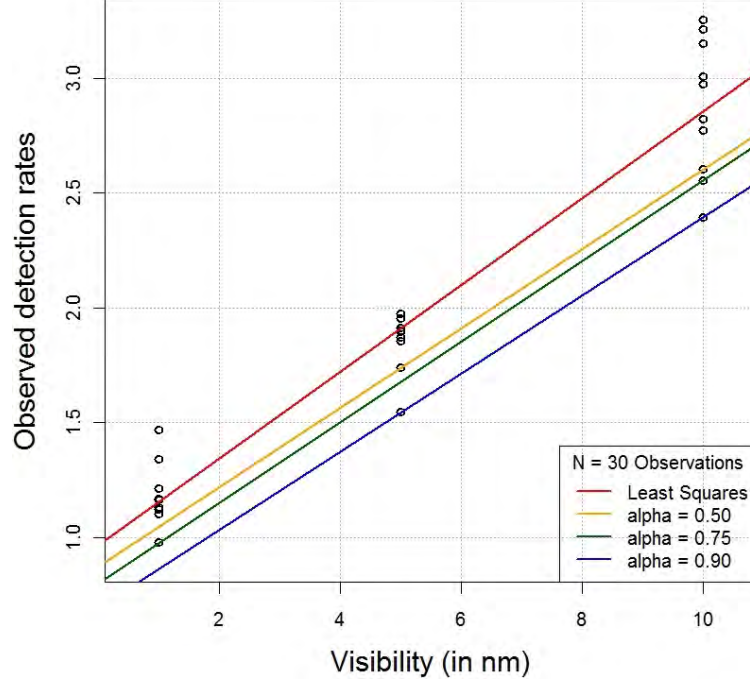


Figure 14. Case C - Comparison between least-squares linear regression and RP(0.25, 0.75), for different  $\alpha$ .

Figure 14 shows the corresponding linearly approximated functions using least-squares linear regression and RP(0.25, 0.75). We note that the nonstandard regression RP(0.25, 0.75) tends to shift the fitted lines towards the smallest realizations of  $\{r^k\}_{k=1}^{30}$  as the probability level  $\alpha$  increases, therefore underestimating the detection rates and returning more conservative regression coefficients. Figure 15 shows exactly the opposite because the weights are the complements of the ones used in Figure 14. This implies that the decision maker should be careful defining the weights  $w_1$  and  $w_2$ .

After obtaining the regression coefficients, we replace the objective function by the approximated one in  $\widehat{\text{SP}}$ , as described in Section IV.C. Table 8 presents the optimal values for Case C. We observe that Case C results in larger optimal values than

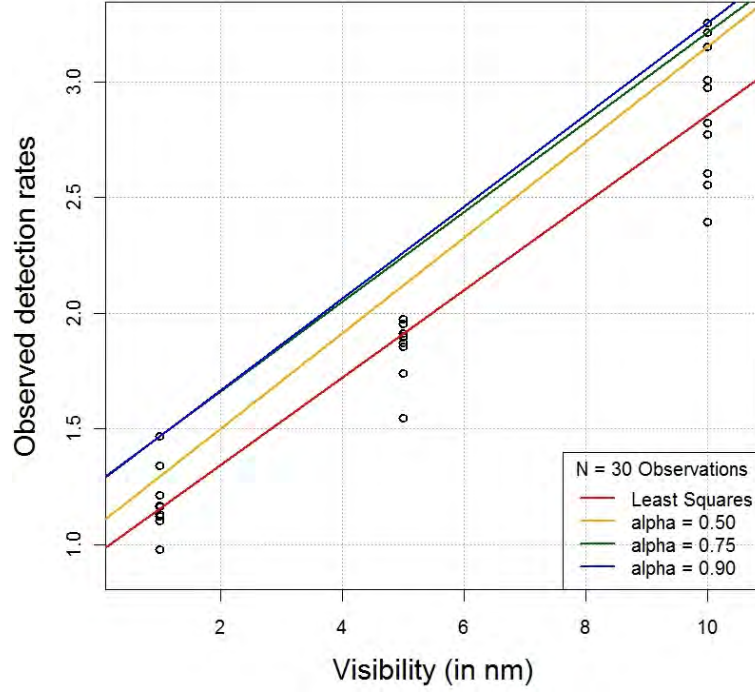


Figure 15. Case C - Comparison between least-squares linear regression and RP(0.75, 0.25), for different  $\alpha$ .

the previous cases, since the nonstandard regression generates conservative estimates of the detection rate for  $\widehat{SP}$ .

$\rho$	Visibility	SP- $S_\alpha$			$\widehat{SP}$
		Case A ( $R_{c,t}$ constant $\forall c, t$ )	Case A ( $R_{c,t}$ varies between $c$ and $t$ )	Case B (random $R_{c,t}$ constant $\forall c, t$ )	Case C (random $R_{c,t}$ constant $\forall c, t$ )
0.6	Good	0.6939	0.6947	0.6887	0.8465
0.4		0.7634	0.7646	0.7539	0.8831
0.2		0.7962	0.7974	0.7949	0.9007

Table 8. Comparison of optimal objective function values between Cases, using superquantile with  $\alpha = 0.75$  (Good visibility,  $w_1 = 0.25, w_2 = 0.75$ ), where the objective function is the risk measure of the probability of missing the target.

For the situation where the regression coefficients are obtained by using the

least-squares linear regression, we use  $\alpha = 0$  in  $\widehat{SP}$ . Figure 16 shows that the optimal

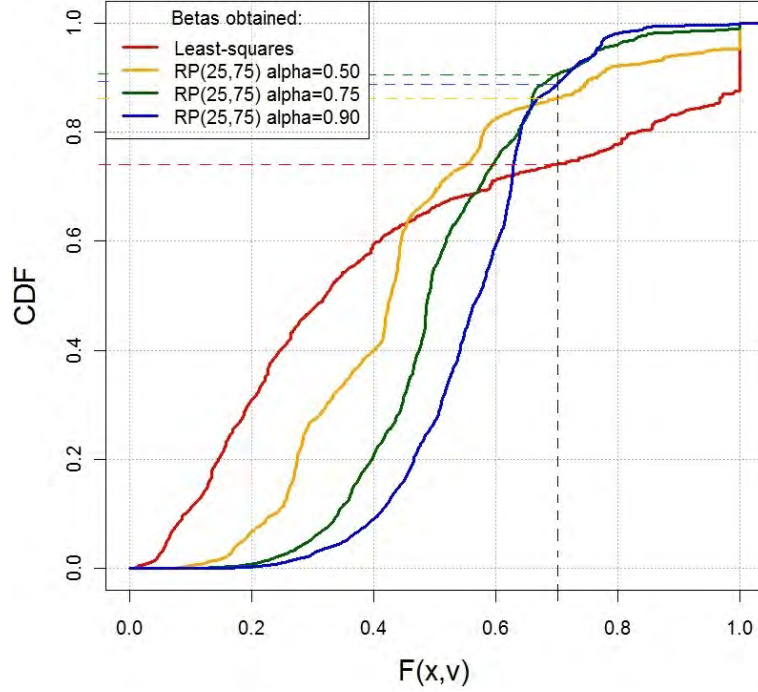


Figure 16. Case C - Cumulative distribution functions for  $F(x, V)$  given optimal search plan  $x$  and random detection rate. Comparison between least-squares and RP(0.25, 0.75) regression coefficients (Good visibility,  $\rho = 0.6$ ).

search plan for the least-squares performs better for thresholds for  $F(x, V)$  up to 45%. Then using  $\alpha = 0.50$  is the best solution until a threshold of 66%. After a 66% probability of missing the target it is almost indistinguishable which  $\alpha$  to choose, therefore one consideration should be the run times of  $\widehat{SP}$  for each value  $\alpha$ . From all the numerical examples, we note that the larger the  $\alpha$ , the longer the run time of  $\widehat{SP}$ . From Figure 16 we observe that if we establish a goal of preventing probabilities of missing the target of 70% or higher, we have 92% chance of doing better than the goal if we rely on RP(0.25, 0.75) with  $\alpha = 0.75$  to obtain the regression coefficients and use them in  $\widehat{SP}$ , while only 74% chance if we use the least-squares regression coefficients in  $\widehat{SP}$ .

Figures 17 and 18 show the same comparison for fair and poor visibility conditions. We notice that the optimal search plans obtained using least-squares regression

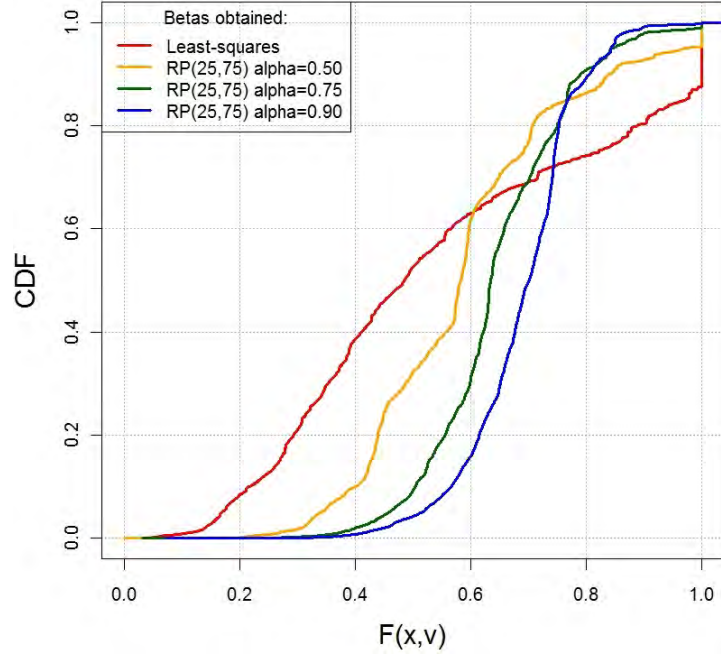


Figure 17. Case C - Cumulative distribution functions for  $F(x, V)$  given optimal search plan  $x$  and random detection rate. Comparison between least-squares and RP(0.25, 0.75) regression coefficients (Fair visibility,  $\rho = 0.6$ ).

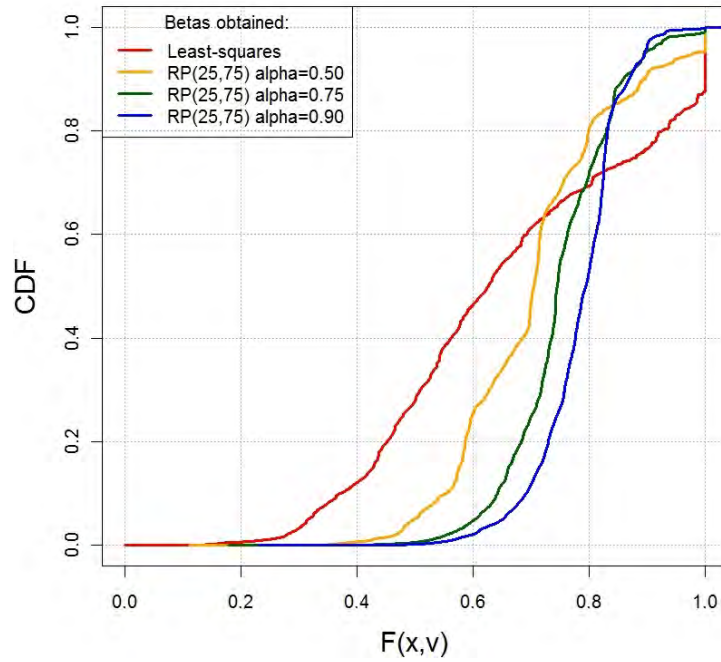


Figure 18. Case C - Cumulative distribution functions for  $F(x, V)$  given optimal search plan  $x$  and random detection rate. Comparison between least-squares and RP(0.25, 0.75) regression coefficients (Poor visibility,  $\rho = 0.6$ ).

perform better for smaller thresholds of  $F(x, V)$ . However towards the tail of the cumulative distribution function of the probability of missing the target, using larger values of  $\alpha$  gives higher chance of performing better than the established threshold.



## V. CONCLUSIONS

In this thesis, we discuss a type of stochastic optimization problems where the decision maker does not have complete information concerning the objective function. Specifically, we consider a discrete time-and-space search optimization problem (SP). We want to route search assets in order to detect a moving target such as a drug smuggler in an area of operations.

We assume that each searcher is equipped with one imperfect sensor, which is not subject to false-positive detections. The searchers are all alike, equipped with the same type of sensor, and they are allowed to divide their effort across multiple cells in arbitrarily small portions. The goal is to determine a search plan such that the probability of missing the target is minimized, without having the full knowledge of the two sources of uncertainty: the target location and the sensor performance. We assume that the target paths are probabilistically known via intelligence reports; for example from data acquired by AIS (Automatic Identification System). We use a Markov chain model to generate the paths with three stationary probabilities. We assume that the sensor performance depends on environmental conditions, and we use visibility to represent these conditions.

We formulate the objective function of SP in terms of a risk measure of a parameterized random variable and consider three cases involving various degrees of knowledge about the sensor performance. In all cases, we consider both the expectation and superquantile risk measures. While the expectation results in an objective function representing the probability of missing the target, the superquantile gives rise to more conservative search plans that perform reasonably well even under exceptional circumstances.

In the first two cases, we consider a random detection rate with known probability distribution. For Case A we present two possible situations, the first for a deterministic and constant detection rate over all cells and time periods, and the sec-

ond for a deterministic detection rate that varies between cells and time periods. The numerical examples show that the objective function values for a varying detection rate are smaller than for a constant one, since the search includes the cells with higher detection rates. One example shows that if the goal is to prevent getting probabilities of missing the target of 70% or higher, then the decision maker should use the superquantile risk measure with  $\alpha = 0.90$ . In that situation there is a 92% chance of obtaining better mission outcomes than the goal, while there is only 81% chance of getting better than the goal when using the expectation. This difference might be critical when choosing a search plan and is more significant for faster moving targets.

For Case B the detection rate is random but constant for all cells and time periods. The results obtained for Case B are close to the results from Case A. We show different comparisons between cumulative distribution functions that demonstrate how a decision maker may benefit from different risk measures depending on the threshold for the probability of missing the target.

In the third case, the distribution of the sensor detection rate is unknown and we approximate the detection rate by a linear combination of well-known factors using a nonstandard regression. We use visibility conditions as the factor in the numerical examples. Based on a table of observed sensor performances, that regression minimizes the error induced by the approximation in some sense. We compare the obtained coefficients with the ones resulting from a least-squares linear regression model. We show that the coefficients obtained from the nonstandard regression are more conservative than the ones from least-squares, by underestimating the detection rate. Using the approximations in the objective function of SP, we obtain an approximate problem that we solve for situations involving stochastic information about the visibility conditions. The nonstandard regression utilizes all observed sensor performances and we obtain worse results than the ones from Cases A and B, since there is only probabilistic information about what visibility conditions to expect during the mission. However we show that the optimal search plan performs better in ex-

exceptional circumstances if obtained by using the nonstandard regression coefficients instead of the ones from least-squares linear regression in the approximate problem. We show that for good visibility we have an 18% higher chance of preventing probabilities of missing the target of 70% or higher if we use the search plan that relies on nonstandard regression coefficients with  $\alpha = 0.75$  in SP.

In all cases, the numerical results show that the superquantile spreads the searchers over the area of operations, covering a larger area with smaller searcher fractions than when using expectation. This risk measure handles exceptional target paths more effectively.

## **A. FUTURE RESEARCH**

Future research could generalize the bounds between the risk measure of the true random function and the risk measure of the approximated one beyond an exponential of a sum of linear functions. This generalization could lead to new nonstandard regression techniques.

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